SIGNAL PROCESSING ISSUES IN DIFFRACTION AND HOLOGRAPHIC 3DTV

Levent Onural and Haldun M. Ozaktas

Department of Electrical Engineering, Bilkent University
TR-06800 Bilkent, Ankara, Turkey
www.3dtv-research.org

ABSTRACT

Image capture and image display will most likely be decoupled in future 3DTV systems. For this reason, as well as the need to convert abstract representations to display driver signals, and the need to explicitly consider diffraction and propagation effects, it is expected that signal processing issues will play a fundamental role in achieving 3DTV operation. Since diffraction between two parallel planes is equivalent to a 2D linear shift-invariant system, various signal processing techniques play an important role. Diffraction between tilted planes can also be modeled as a relatively simple system, leading to efficient discrete computations. Two fundamental problems are digital computation of the optical field due to a 3D object, and finding the driver signals for a given optical device so as to generate the desired optical field in space. The discretization of optical signals leads to several interesting issues; for example, it is possible to violate the Nyquist rate while sampling, but still maintain full reconstruction. The fractional Fourier transform is another signal processing tool which finds application in optical wave propagation.

1. INTRODUCTION

Regardless of the algorithmic, representational, and technological choices made for the acquisition, transmission, and display of three-dimensional (3D) visual signals, optics is expected to play a more important role in holographic three-dimensional television (3DTV), than it does in conventional display technologies such as cathode ray tubes and liquid crystal displays or cinematic projection. This is because the creation of a three-dimensional image, or the illusion of it, depends on the manipulation of light for the purpose of synthesizing desired spatial light distributions. The analyses of the underlying processes will almost certainly involve explicit consideration of diffraction and related phenomena.

The image capture and image display steps will most likely be decoupled in future 3DTV systems: The captured 3D scene and object will be stored in convenient forms, and then the viewer at the display-end will access the abstract 3D information in an interactive fashion, with the abstract data being converted to signals that will drive the high-quality display.

Therefore, due to this decoupled approach and the need to convert abstract representations to driver signals, as well as the need to explicitly consider diffraction and propagation effects, it is expected that signal processing issues will play a fundamental role in achieving 3DTV operation. The purpose of this paper is to identify some of the key signal processing issues in holographic 3DTV. The formulation of diffraction phenomena, forward and the inverse problems in holographic 3DTV, discretization issues, and the use of the fractional Fourier transform as a tool for analyzing optical phenomena are the main topics covered in this paper.

2. REVIEW OF DIFFRACTION FROM A SYSTEMS POINT OF VIEW

It is well known that scalar monochromatic diffraction in homogeneous media can be exactly represented as an all-pass linear shift-invariant (LSI) system [1]. Based on the plane-wave decomposition of optical propagation one can write

$$\psi_{2D_Z}(x,y) \triangleq \psi(x,y,Z) = \frac{1}{4\pi^2} \iint_{k_x^2+k_y^2 \le k^2} A(k_x,k_y) \exp\left[jZ(k^2-k_x^2-k_y^2)^{1/2}\right] \exp\left[j(k_x x + k_y y)\right] dk_x dk_y \quad , \tag{1}$$

where $\psi(x,y,z)$ is the 3D coherent optical field, and $\psi_{2D_Z}(x,y)$ is its 2D cross section at $z = Z$. Therefore, $\psi_{2D_Z}(x,y)$ is the diffraction pattern over a planar 2D surface, due to an object transparency mask $a(x,y)$ located at $z = 0$. $A(k_x,k_y)$ is the Fourier transform of $a(x,y)$. k_x and k_y are the spatial frequencies along the x and y axes respectively. The monochromatic light wavelength is λ and $k = 2\pi/\lambda$. Therefore, the transfer function of the 2D LSI system is $\exp\left[jZ(k^2-k_x^2-k_y^2)^{1/2}\right]$, for $k_x^2+k_y^2 \le k^2$. Surprisingly, it is very unlikely to see the inverse Fourier transform of this function in texts or tables. However, it is proven by Sherman [2] that the inverse Fourier transform (i.e., the impulse response of the system representing the diffraction of light) is the kernel of the well-known Rayleigh-Sommerfeld solution [1]. For distances which are large compared to the wavelength the impulse response reduces to the kernel related to the spherical propagation of light out of a point source:

$$h_Z(x,y) \approx \frac{Z}{j\lambda(x^2+y^2+Z^2)} \exp\left[j\frac{2\pi}{\lambda}\left(x^2+y^2+Z^2\right)^{1/2}\right] \tag{2}$$

which is known as the Rayleigh-Sommerfeld diffraction formula [1].

Under the paraxial approximation (i.e., if the angle between the z-axis, and the line connecting a point of interest on the diffraction plane to an active point on the object mask is small) the impulse response and the associated transfer functions become [1]:

$$h_{Z_1}(x,y) = \frac{1}{j\lambda Z} \exp\left(j\frac{2\pi}{\lambda}Z\right) \exp\left[j\frac{\pi}{\lambda Z}(x^2+y^2)\right] \tag{3}$$

$$H_{Z_1}(k_x,k_y) = \exp\left(j\frac{2\pi}{\lambda}Z\right) \exp\left[-j\frac{\lambda Z}{4\pi}(k_x^2+k_y^2)\right] \tag{4}$$

The paraxial approximation above is also known as the Fresnel or Huygens-Fresnel approximation. The Fresnel approximation represented by the convolution of $a(x,y)$ with the impulse response given above, can be easily converted to a single Fourier transform with pre- and post- multiplications by quadratic phase functions:

$$\begin{aligned}\psi_{2D_Z}(x,y) =& \\ & c\iint a(\xi,\eta) \\ & \qquad \exp\left\{j\frac{\pi}{\lambda Z}\left[(x-\xi)^2+(y-\eta)^2\right]\right\}d\xi d\eta \\ =& c\exp\left[j\frac{\pi}{\lambda Z}(x^2+y^2)\right]\iint a(\xi,\eta) \\ & \exp\left[j\frac{\pi}{\lambda Z}(\xi^2+\eta^2)\right]\exp\left[j\frac{2\pi}{\lambda Z}(x\xi+y\eta)\right]d\xi d\eta\end{aligned} \tag{5}$$

where the constants are combined into the new constant c.

At very large distances (small objects), further approximation yields the Fourier diffraction pattern as a quadratic-phase modulated by the Fourier transform of the object [1]:

$$\begin{aligned}\psi_{2D_Z}(x,y) =& \\ & \exp\left[j\frac{\pi}{\lambda Z}(\xi^2+\eta^2)\right]\iint a(\xi,\eta) \\ & \qquad \exp\left[j\frac{2\pi}{\lambda Z}(x\xi+y\eta)\right]d\xi d\eta\end{aligned} \tag{6}$$

The past decade has witnessed a recognition of the relationship between optical propagation and the fractional Fourier transform (FRT) [3]. The FRT $f_a(x)$ of $f(x)$ is defined as

$$\begin{aligned}f_a(x) = A_{a\pi/2}\int_{-\infty}^{\infty}\exp[i\pi(x^2\cot(a\pi/2)-2xx'\csc(a\pi/2)+ \\ x'^2\cot(a\pi/2))]f(x')\,dx',\end{aligned} \tag{7}$$

where $A_{a\pi/2}$ is a factor depending on a whose exact form is not of importance here. The key result is that relating free-space propagation in the Fresnel approximation (the Fresnel integral or the Fresnel transform), to the fractional Fourier transform [4, 5, 6, 7]. Extensions of this result relate arbitrary linear canonical transforms to the fractional Fourier transform; for instance [8]. Linear canonical transforms are a three-parameter family of integral transforms which are also known as quadratic-phase systems. This family of transforms includes the Fourier and fractional Fourier transforms, simple scaling including the identity and parity operations (corresponding to imaging in optics), chirp multiplication and convolution operations (corresponding to passage through a thin lens and free-space propagation in the Fresnel approximation respectively), and hyperbolic transforms as special cases [9, 10]. Since optical systems consisting of arbitrary concatenations of lenses and section of free space can be modeled as linear canonical transforms, it follows that propagation through such systems, as well as free-space propagation can be viewed as an act of continual fractional transformation. The wave field evolves through fractional Fourier transforms of increasing order as it propagates through free space or the multi-lens system.

Restricting ourselves to one-dimensional notation for simplicity, the output $g(x)$ of a quadratic-phase system is related to its input $f(x)$ through

$$g(x) = \sqrt{\beta}\,e^{-i\pi/4}\int_{-\infty}^{\infty}\exp[i\pi(\alpha x^2-2\beta xx'+\gamma x'^2)]\,f(x')\,dx', \tag{8}$$

where α,β,γ are the three parameters of the system. When all three of these parameters equal $1/\lambda Z$, this expression reduces to the Fresnel integral (within an inconsequential phase factor). The same relationship can also be written in terms of an alternate set of parameters a, M, R as follows:

$$\begin{aligned}g(x) = e^{i\pi x^2/\lambda R}\sqrt{\frac{1}{s^2M}}A_{a\pi/2}\int_{-\infty}^{\infty}\exp\left[\frac{i\pi}{s^2}\left(\frac{x^2}{M^2}\cot(a\pi/2)\right.\right. \\ \left.\left.-2\frac{xx'}{M}\csc(a\pi/2)+x'^2\cot(a\pi/2)\right)\right]f(x')\,dx',\end{aligned} \tag{9}$$

where s is an arbitrary scale factor. This relationship maps a function $s^{-1/2}f(x/s)$ to $\exp(i\pi x^2/\lambda R)\sqrt{1/sM}\,f_a(x/sM)$. That is, $g(x)$ is essentially the ath order fractional Fourier transform of $s^{-1/2}f(x/s)$, scaled by M, and multiplied by a residual quadratic-phase factor. In optics scaling corresponds to magnification of the distribution of light in the transverse direction. The existence of the quadratic-phase factor means that the magnified fractional Fourier transform is observed on a spherical reference surface, rather than on a plane. Comparing Eqs. 8 and 9, we can relate the two sets of parameters as follows:

$$\alpha = \frac{\cot(a\pi/2)}{s^2M^2}+\frac{1}{\lambda R}, \tag{10}$$

$$\beta = \frac{\csc(a\pi/2)}{s^2M}, \tag{11}$$

$$\gamma = \frac{\cot(a\pi/2)}{s^2}. \tag{12}$$

These equations allow us to switch between the two sets of parameters and thus interpret any quadratic-phase integral and thus the wide class of optical systems they represent as fractional Fourier transforms. Since the FRT has a much broader set of properties mirroring those of the ordinary Fourier transform, and is geometrically and numerically much better behaved, formulating the propagation of light through optical systems in terms of the FRT has several advantages. As a special case, when $\alpha=\beta=\gamma=1/\lambda Z$ corresponding to ordinary free-space propagation, we have

$$\tan(a\pi/2) = \frac{\lambda Z}{s^2}, \tag{13}$$

$$M = \sqrt{1+(\lambda Z/s^2)^2}, \tag{14}$$

$$\frac{1}{\lambda R} = \frac{1}{s^4}+\frac{\lambda Z}{1+(\lambda Z/s^2)^2}. \tag{15}$$

To summarize, the expression for scalar diffraction, and its Fresnel and Fourier approximations, the former possibly expressed as a fractional Fourier transform, are of fundamental importance and are key relationships whose manipulation and computation are of importance for the synthesis and/or reconstruction of 3D light fields. Various signal processing techniques for the simulation of diffraction are essential to this end.

These issues become particularly important when one deviates from standard problems. For instance, consider the more difficult problem of diffraction between planes tilted with respect to each other, instead of being parallel as formulated above [11, 12]. Using the plane wave decomposition approach for scalar optical waves, we can form superpositions in 3D space for monochromatic waves. Intersecting such a 3D pattern by two tilted planes, we observe that each 3D plane wave component yields a 2D frequency component over one of the planes, where the corresponding pattern over the other (not necessarily parallel) plane has a different frequency due to the tilt. It is then easy to compute the corresponding amplitude, frequency, and phase component pairs over the two planes of interest. To do so, we start with the representation of the coordinates of the observation (diffraction) plane as

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{b} \tag{16}$$

where $\mathbf{R}$ is a 3D rotation matrix, $\mathbf{b}$ is the 3D translation vector in space, and $\mathbf{x}$ is the coordinate vector $[x\ y\ z]^T$. Therefore, the desired diffraction pattern can be found as

$$g(x,y) = \mathscr{F}^{-1}_{(k_x',k_y')' \to (x,y)} \left\{ \mathscr{F}_{(x,y)\to(k_x,k_y)} \{ f(x,y) \} \Big|_{\mathbf{k}\to\mathbf{R}\mathbf{k}'} H(\mathbf{k}',\mathbf{R},\mathbf{b}) \frac{k_z}{k_z'} \right\} \tag{17}$$

where $\mathscr{F}$ represents the 2D Fourier transform, and its arrowed subscripts denote the variables of the pre- and post-Fourier transform domains. The function $H(\mathbf{k}',\mathbf{R},\mathbf{b})$ provides the kernel of the corresponding system, represented by

$$H(\mathbf{k}',\mathbf{R},\mathbf{b}) = e^{j\mathbf{k}'^T(\mathbf{R}^T\mathbf{b})}. \tag{18}$$

Furthermore, the two 2D frequency vectors (k_x, k_y), (k_x', k_y') at the object and the observation planes, respectively, are related by $\mathbf{k} = \mathbf{R}\mathbf{k}'$.

3. FORWARD AND INVERSE PROBLEMS IN HOLOGRAPHIC 3DTV

The two fundamental problems in holographic 3DTV are what we will refer to as the forward and the inverse problems.

The forward problem is the computation of the light field distribution which arises over the entire 3D space, given an abstract 3D structure with specified shape and texture. This is the light field which we will desire to create at the display end, but in order to do so, we must first compute what it is. This is a considerably more difficult problem compared to the classical textbook problems outlined in the previous section, because the 3D structure is not a simple plane, but consists of a complex structure of opaque or transparent or semi-transparent surfaces.

Once this field is determined, physical devices will be used to create this field at the display end, which will then propagate in 3D space before reaching the viewer. These physical devices impose many constraints as a consequence of their particular characteristics and limitations. Therefore, the 3D light distribution we are able to generate these devices might not be able to exactly match the desired 3D field computed by solving the forward problem outlined above. Furthermore, the relationship between the electronic or other forms of driving signals of these devices, to the generated 3D light field may not be straightforward. Therefore, given a physical device, like a specific spatial light modulator [13], or an acousto-optic element [14], finding the driving signals to get the best approximation to the given desired 3D light field is an intriguing and challenging inverse problem.

4. SAMPLING ISSUES IN DIFFRACTION

Discretization and subsequent quantization are unavoidable in digital computations within the context of the forward or the inverse problems outlined above. Furthermore, digital versions of the systems discussed in Section 2 require discretization of the patterns and the presented kernels. Due to the specific nature of these diffraction kernels, naive attempts at discretization will usually not lead to satisfactory results. It is of paramount importance to understand the exact effects of sampling on these special types of systems and their interpretation. This allows much more efficient sampling schemes than naive approaches based on straightforward application of Nyquist-Shannon's theorem would achieve. We discuss a number of related issues in what follows.

Looking back to the exact scalar diffraction expression given by Eq. 1, we see that the transfer function is band-limited to a circle. However, the Fresnel diffraction kernel, given by Eq. 4, is neither band nor space limited. Despite the tendency to directly apply standard Nyquist-Shannon sampling theory, as shown in [15] [16], employing band-limited sampling and associated sinc interpolation for such expressions results in unnecessary and extremely redundant sampling rates. Instead, using the concept of α-Fresnel limitedness, which fits naturally to most diffraction cases under the Fresnel approximation, the sampling rates can be significantly reduced below the Nyquist rate, but still yields perfect reconstruction of the underlying analog functions. For example, it is shown in [15] that such a sampling generates modulated and shifted replicas of the original space-limited object as

$$\psi_R(\mathbf{x}) = \sum_{\mathbf{k}} c_{\mathbf{k}} f(\mathbf{x} - \frac{\lambda z}{2\pi}\mathbf{U}\mathbf{k}) \exp\left(j\mathbf{k}^T\mathbf{U}^T\mathbf{x}\right), \tag{19}$$

where $c_{\mathbf{k}}$ is the indexed set of related coefficients, and the periodicity matrix $\mathbf{U}$ is related to the sampling matrix, $\mathbf{V}$ by $\mathbf{U} = 2\pi\mathbf{V}^{-T}$. Bold face vectors $\mathbf{x}$, and $\mathbf{k}$ are the vectors, $[x\ y]^T$ and $[k_x\ k_y]^T$, respectively. Therefore, full recovery is still possible even if the Nyquist criteria is severely violated by simply windowing the desired space-limited object by leaving the replicas out.

Applying the sub-Nyquist sampling to the exact diffraction expression given in Eq. 1 reveals very interesting results: the modulated and shifted replicas of the sub-Nyquist sampled Fresnel diffraction now get dispersed, and this dispersion may result in a favorable situation during reconstruction by naturally reducing the visibility of the replicas. It is easy to understand this effect if we observe that the discrete sampling function $p(x,y)$ at the diffraction plane can be written as

$$p(x,y) = \sum_{\mathbf{m}} \delta(\mathbf{x} - \mathbf{V}\mathbf{m}) = \mathscr{F}^{-1}\left\{P(k_x,k_y)\right\} = \frac{1}{|\det \mathbf{V}|} \sum_{\mathbf{m}} \exp\left[j(k_{x,\mathbf{m}} x + k_{y,\mathbf{m}} y)\right] \tag{20}$$

through its inverse Fourier transform, where $[k_{x,\mathbf{m}}\ k_{y,\mathbf{m}}]^T = 2\pi\mathbf{V}^{-T}\mathbf{m}$. But this can be interpreted as the 2D cross section, at the diffraction plane, of a bunch of superposed discrete-direction 3D plane waves. Therefore, this sampling operation can be interpreted as the parallel reconstruction from the diffraction pattern, by a number of ”illuminating” plane waves each propagating at a different direction, where the superposition forms the sampling grid right at the diffraction plane. One of those reconstructions corresponds to the ”desired” reconstruction, whereas others give the reconstruction by higher diffraction orders. Even though those diffraction orders generate the shifted-modulated replicas of the original object for the Fresnel case (Eq. 19), they yield dispersed replicas when the exact formulation is used.

Yet another interesting sampling related result is presented in [17]: it is shown that for some periodic input patterns, the exact *continuous* Fresnel diffraction pattern at given distances can be computed by discrete signal processing techniques (DFT).

The fractional Fourier transform formulation provides an integral approach to handling the sampling issue. Referring to Eq. 7, we again observe that naive application of the Nyquist-Shannon approach may require very large sampling rates due to the highly oscillatory nature of the kernel. However, by careful consideration of sampling issues, it is possible to accurately and efficiently compute this integral with a number of samples close to the space-bandwidth product of $f(x)$ [18].

It is also interesting to note that wavelet structures, and their applications to diffraction problems [19] provide rich signal processing tools for approaching and efficiently solving problems of a nature outlined above.

5. CONCLUSION

The diffraction in general, and the associated issues related to holographic 3DTV in particular naturally have a close link with various signal processing topics. The classical linear shift invariant system structure for the diffraction between two parallel planes has been well known and utilized. However, applications of signals and systems approaches to other issues like the 3D volume diffraction patterns, diffraction pattern between arbitrary tilted planes are rather new. The natural link between the fractional Fourier transform, and the interesting results associated with the sampling of the diffraction field are promising for further interesting applications.

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REFERENCES

[1] J. W. Goodman. *Introduction to Fourier Optics*. McGraw-Hill, New York, second edition, 1996.

[2] G. C. Sherman. Application of the convolution theorem to rayleigh’s integral formulas. *J Opt Soc Am*, 57:546–547, 1967.

[3] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay. *The Fractional Fourier Transform with Applications in Optics and Signal Processing*. Wiley, New York, 2001.

[4] H. M. Ozaktas and D. Mendlovic. Fractional Fourier transform as a tool for analyzing beam propagation and spherical mirror resonators. *Opt Lett*, 19:1678–1680, 1994.

[5] P. Pellat-Finet. Fresnel diffraction and the fractional-order Fourier transform. *Opt Lett*, 19:1388–1390, 1994.

[6] F. Gori, M. Santarsiero, and V. Bagini. Fractional Fourier transform and Fresnel transform. *Atti della Fondazione Giorgio Ronchi*, IL:387–390, 1994.

[7] H. M. Ozaktas and D. Mendlovic. Fractional Fourier optics. *J Opt Soc Am A*, 12:743–751, 1995.

[8] H. M. Ozaktas and M. F. Erden. Relationships among ray optical, gaussian beam, and fractional Fourier transform descriptions of first-order optical systems. *Opt Commun*, 143:75–86, 1997.

[9] K. B. Wolf. Construction and properties of canonical transforms. In *Integral Transforms in Science and Engineering*. Plenum Press, New York, 1979. Chapter 9.

[10] M. J. Bastiaans. Wigner distribution function and its application to first-order optics. *J Opt Soc Am*, 69:1710–1716, 1979.

[11] N. Delen and B. Hooker. Free-space beam propagation between arbitrarily oriented planes based on full diffraction theory: A fast Fourier transform approach. *J Opt Soc Am A*, 15:857–867, 1998.

[12] G. B. Esmer and L. Onural. Simulation of scalar optical diffraction between arbitrarily oriented planes. In *First International Symposium on Control, Communications and Signal Processing*, 2004.

[13] S. Esener, J. L. Horner, and K. M. Johnson. Spatial light modulators: Introduction to the special issue. *Appl Opt*, 31:3879, 1992.

[14] L. Onural, G. Bozdağı, and A. Atalar. New high-resolution display device for holographic three-dimensional video: Principles and simulations. *Opt Eng*, 33:835–844, 1994.

[15] L. Onural. Sampling of the diffraction field. *Appl Opt*, 39:5929–5935, 2000.

[16] F. Gori. Fesnel transform and sampling theorem. *Opt Commun*, 39:293–297, 1981.

[17] L. Onural. Some mathematical properties of the uniformly sampled quadratic phase function and associated issues in Fresnel diffraction simulations. *Opt Eng*, 43:2557–2563, 2004.

[18] H. M. Ozaktas, O. Arikan, M. A. Kutay, and G. Bozdağı. Digital computation of the fractional Fourier transform. *IEEE Trans Signal Process*, 44:2141–2150, 1996.

[19] L. Onural and M. Kocatepe. Family of scaling chirp functions, diffraction, and holography. *IEEE Trans Signal Process*, 43:1568–1578, 1995.